Math 1512 - Exam 2 Study Guide

Daniel Havens
dwhavens@unm.edu

March 8, 2023

Contents

Summary and Disclaimer

This is a study guide for the second exam for math 1512 at the University of New Mexico (Calculus I). The exam covers sections 2.6-2.9, 3.1-3.5, and 3.7 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material previously covered in calculus 1, as well as the material from college algebra and trigonometry. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

The focus of this exam is on applications of derivatives. In particular, we focus on related rates, optimization, and graphing. The first two are different flavors of word problems, while the last expands on what you will know from a college algebra (or similar) course.

Our first method is implicit differentiation, which tells us how to find derivatives of formulas which are too messy to rearrange for just y.

Implicit Differentiation

If we have a function of the form

$$f(x)g(y) = 0$$

then we can take the derivative of both sides, using the chain rule and product rule on the left, to get

$$f'(x)g(y) + y'g'(y)f(x) = 0.$$

Then we can rearrange for y'.

Implicit differentiation is quite confusing without working some examples, so make sure to check the worked examples section for a few of those.

Next up are related rates problems. These are an application of derivatives, so there is more a list of steps to go through than an actual laid-out process.

Related Rates

1. Draw a picture of the situation.
2. Come up with an equation describing your situation.
3. Take the derivative of the equation.
4. Plug in all of the information you have. Find out what you are still missing.
5. Use your picture and other equations to find what you are missing.
6. Solve for what you are asked to find.

Next, we would like to approximate functions nicely, given a base-point.

Approximating Functions Linearly

To approximate a function linearly given a base-point (x_0, y_0), we have the following formula, given by the equation of the tangent line at the indicated point:

$$f(x) \approx f'(x_0)(x - x_0) + y_0$$

Often when approxmating a function linearly, we are not given the y_0 or the x_0. Instead, we have to extrapolate what the x_0 is, and find the y_0 from $f(x_0) = y_0$. It may help to note that the phrase "when x is small" really just means that $x_0 = 0$.

Next, we have a similar formula for approximating changes in functions.

Approximating Changes in Functions

To approximate the change of a function at a given base-point (x_0, y_0), we have the following formula:

$$\Delta f \approx f'(x_0)\Delta x.$$

where Δf is how much the function changes, and Δx is how much the input changed.

For convenience, we combine the first derivative test and the second derivative test into one test, the "first and second derivative test". This tells us when a function has a minimum, a maximum, or an inflection point.

First and Second Derivative Test

If $f''(x_0) = 0$, then x_0 is an inflection point.
If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a local maximum.
If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum.

Next, we use this to find the minimum and maximum of a function on a closed interval:

Minima and Maxima on Closed Intervals

To find the minimum or maximum of a function $f(x)$ on the interval $[a, b]$, first, do the first derivative test to find the local minima and maxima, and see which are in your interval. Then, check the values of $f(a)$ and $f(b)$. The absolute minimum is the smallest of all of these, and the absolute maximum is the biggest of all of these.

Graphing consists more of a set of steps. Here are mine, but be aware that everybody has their own work-flow for these kinds of problems. If at any point you say "I can't do this step" for one reason or another, just move on to the next step. The goal is to create the most complete graph possible, not a perfect graph.

Graphing

1. Find the zeroes of your function.
2. Find $f'(x)$, and find the zeroes of it.
3. Find $f''(x)$ and find the zeroes of it.
4. Use the above information to find the minima, maxima, and inflection points of the function.
5. Use the above information to find where your function is increasing, decreasing, concave up, and concave down.
6. Find the asymptotes of your function.
7. Find the endpoint behavior by evaluating the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
8. Use everything above to graph the function as accurately as possible.

Optimization, like related rates, is an application of the above methods. Because of this, there is more of a list than an explicit formula. Here is the list as I like to work through it. Be aware that your instructor may have taught you something slightly different, and that is perfectly fine.

Optimization

1. Turn your word problem into a set of equations. Make sure you include any restrictions (these are often numbers not being negative).
2. Determine what you are trying to optimize.
3. Rearrange your other equations to get all of your variables in terms of one variable (for instance, if you are using x and have a y in your equation, solve for y in terms of x).
4. Substitute the above into your equation you are optimizing (that way your equation only has one variable).
5. Find the minimum or maximum of the function you are optimizing using calculus.
6. Use this to find all of the information needed

Worked Examples

We will now work through some examples. We start with an example on implicit differentiation.

Example: Find y' if $\cos(x) + \sin(y) = 43$.
We begin by taking the derivative of both sides. This gives us

$$-\sin(x) + y' \sin(y) = 0.$$

And rearranging gives us that $y' \cos(y) = \sin(x)$, or $y' = \frac{\sin(x)}{\cos(y)}$.

Next, we work an example of linear approximation.

Example: Approximate the function $f(x) = x^3 + 2x^2 + x + 10^{15}$ for small x.
Using our formula, we have that $x_0 = 0$, so $y_0 = f(x_0) = 10^{15}$. So, taking the derivative gives us $f'(x) = 3x^2 + 4x + 1$, and $f'(x_0) = 1$. So, our linear approximation is given by the line $y - 10^{15} = 1(x - 0)$, or $y = x + 10^{15}$. So:

$$x^3 + 2x^2 + x + 10^{15} \approx x + 10^{15}$$

for small x.

As this material is more learned by practice than examples, I have added more practice problems to make up for the lack of examples.

However, more example problems are available on request.

Practice Problems

These practice problems are separate from the unsolved problems. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as the unsolved questions. They also include worked solutions, unlike the unsolved questions section.

1. Find y' if $\sin(y)\cos(x) = 3y$.
2. Find the maximum value of $\sin(x) + x$ on the interval $[0, 2\pi]$.
3. Approximate the change in $\cos(\cos(x))$ for small x.
4. Find two numbers whose sum is 10 and whose product is maximal.

If it is requested, I will add more practice problems.

Practice Problem Solutions

1. Find y' if $\sin(y)\cos(x) = 3y$.

   Solution: We take the derivative of both sides. This gives us
   $$y'\cos(y)\cos(x) - \sin(y)\sin(x) = 3y'.$$
   So, we have that
   $$-\sin(y)\sin(x) = 3y' - y'\cos(y)\cos(x).$$
   So,
   $$y' = -\frac{\sin(x)\sin(y)}{3-\cos(y)\cos(x)}.$$

2. Find the maximum value of $\sin(x) + x$ on the interval $[0, 2\pi]$.

   Solution: We take the derivative. This gives us $\cos(x) + 1$. Setting this to zero gives us $\cos(x) + 1 = 0$ or $\cos(x) = -1$. So, $x = \pi$. Checking all of our points, we have $f(\pi) = \pi + 1$, $f(0) = 0$, and $f(2\pi) = 2\pi$. So, f has a minimum of 0 at $x = 0$, and a maximum of 2π at $x = 2\pi$.

3. Approximate the change in $\cos(\cos(x))$ for small x.

   Solution: Since we have "small x", we have that $x_0 = 0$. So, taking the derivative of $\cos(\cos(x))$ gives us $\sin(x)\sin(\cos(x))$. Plugging in $x_0 = 0$ gives us $\sin(0)\sin(\cos(0)) = 0 \cdot 1 = 0$. So, our approximation is $\Delta\cos(\cos(x)) \approx 0 \cdot \Delta x = 0$.

4. Find two numbers whose sum is 10 and whose product is maximal.

   Solution: Call our two numbers x and y. Our two equations are $P = xy$, the product, and $x + y = 10$. Rearranging the second gives us $y = 10 - x$, so $P = x(10 - x)$, or $P = -x^2 + 10x$. So, $P' = -2x + 10$, and setting this to zero and solving gives us $-2x + 10 = 0$, or $2x = 10$. So, $x = 5$ gives the maximum. Solving for y, we get that $y = 10 - 5 = 5$, so our two numbers are 5 and 5.

Unsolved Questions

Here is a list of 20 unsolved questions which I feel are of similar difficulty to what might be asked of you on an exam.

1. Find y' if $\cos(x)\sin(y) = 4x$.
2. Find y' if $\tan(y) = \tan(x)$.
3. Find y' if $\sqrt{x^2y^2+3} = \cos(4x)$
4. Find y' if $xy^2 + y = 1$
5. Find the maximum value of $x^5 + x + 3$ on the interval $[5, 7]$.
6. Find the maximum value of $\sin(x) - x\cos(x)$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
7. Find the maximum value of $\cos(x) + x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
8. Find the maximum value of $\cos(x) + 2x$ on the interval $[-10, 10]$.
9. Approximate $\sin(x)$ for small x.
10. Approximate $\sqrt[5]{x}$ for x close to 1.
11. Approximate the change in $x\sqrt[3]{x+27}$ for small x.
12. Approximate the change in $\frac{\tan(\cos(x))}{1+x^3}$ for small x.
13. Graph the function $\frac{\sin(x)}{x}$.
14. Graph the function $\frac{x^5-x^3}{x^3}$.
15. Graph the function $x^{\frac{5}{2}} - 10x^{\frac{3}{2}}$.
16. Graph the function $\frac{1}{x} + 3x + x^2$.
17. Graph the function $x\sin(x)$.
18. Graph the function $\cos(x) + x$.
19. We are going to fence in a rectangular field and have 50 feet of material to construct the fence. Determine the dimensions of the field that will enclose the maximum area.
20. Air is being pumped into a spherical balloon at a rate of 4 $\frac{\text{cm}^3}{\text{min}}$. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 16cm.